

CHAPTER 7:

SECOND ORDER DERIVATIVES

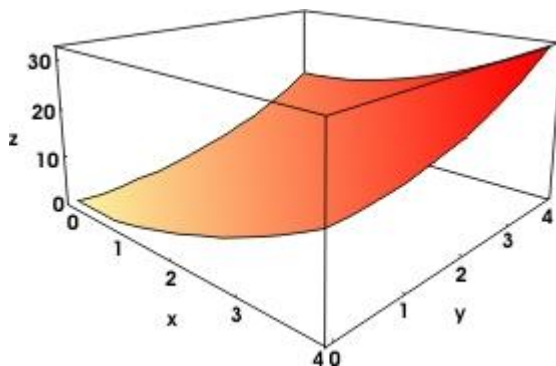
7.1 THE SECOND ORDER DERIVATE $f_{xx}(x, y)$

We have already established that $f_x(x, y)$ represents the slope in the x direction. If the derivative of $f_x(x, y)$ with respect to x is taken, the result is:

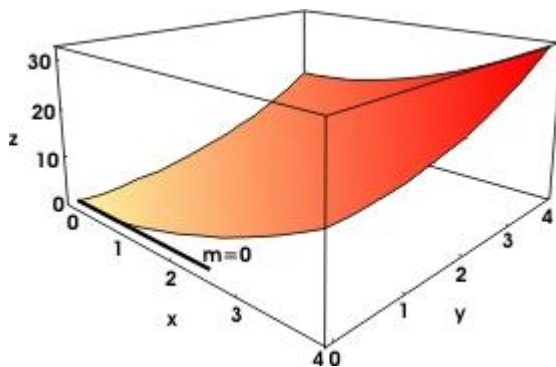
$$\frac{\partial f_x(x, y)}{\partial x} \approx \frac{\Delta f_x(x, y)}{\Delta x} \approx \frac{\Delta m_x}{\Delta x}$$

Geometrically, this is the rate at which the slope in the x direction is changing as we move in the x direction.

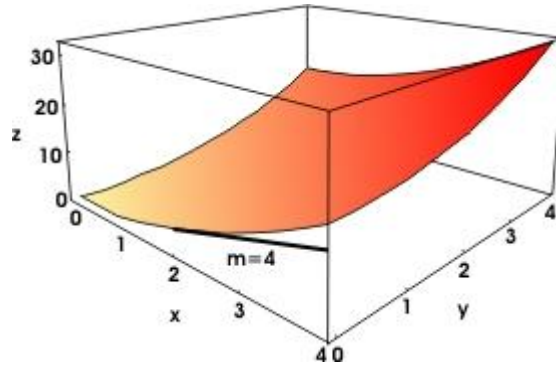
To visualize this geometrically, we can start by observing the graph of $f(x, y) = x^2 + y^2$ shown in the graph below.



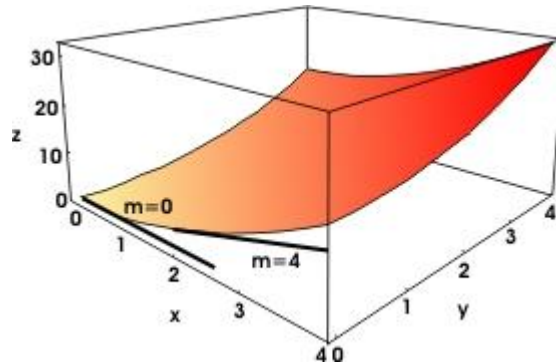
By taking the partial derivative with respect to x and evaluating it at the appropriate point, we can conclude that at $(0, 0, 0)$ the slope in the x direction is 0.



By taking the partial derivative with respect to x and evaluating it at the appropriate point, we can conclude that at $(2, 0, 4)$, the slope in the x direction is 4.



Correspondingly, the change in $f_x(x, y)$ is 4 as we move 2 units in the x direction.



Hence, we can conclude that

$$\frac{\partial f_x(x, y)}{\partial x} \approx \frac{\Delta f_x(x, y)}{\Delta x} \approx \frac{\Delta m_x}{\Delta x} = \frac{4}{2} = 2$$

7.2 THE SECOND ORDER DERIVATE $f_{yy}(x, y)$

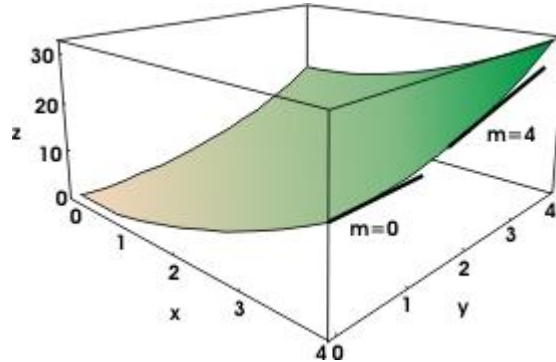
We have already established that $f_y(x, y)$ represents the slope in the y direction. If the derivative of $f_y(x, y)$ with respect to y is taken, the result is:

$$\frac{\partial f_y(x, y)}{\partial y} \approx \frac{\Delta f_y(x, y)}{\Delta y} \approx \frac{\Delta m_y}{\Delta y}$$

Geometrically, this is the rate at which the slope in the y direction is changing as we move in the y direction.

To visualize this geometrically, we can follow the same procedure we followed with $f_{xx}(x, y)$ and the surface $f(x, y) = x^2 + y^2$. By taking the partial derivative with respect to y and evaluating it at the appropriate point, we can conclude that $(4, 0, 16)$, the slope in the y direction is 0. By taking

the partial derivative with respect to y and evaluating it at the appropriate point, we can conclude that at $(4, 2, 20)$, the slope in the y direction is 4.



Correspondingly, the change in $f_y(x, y)$ is 4 as we move 2 units in the y direction. Hence we can conclude that

$$\frac{\partial f_y(x, y)}{\partial y} \approx \frac{\Delta f_y(x, y)}{\Delta y} \approx \frac{\Delta m_y}{\Delta y} = \frac{4}{2} = 2$$

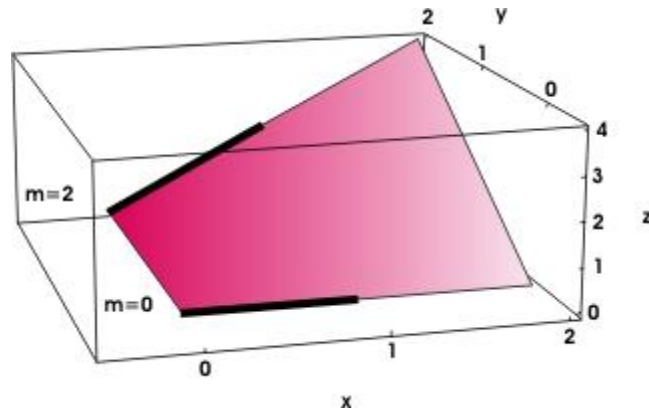
7.3 THE SECOND ORDER DERIVATIVE $f_{xy}(x, y)$

We have already established that $f_x(x, y)$ represents the slope in the x direction. If the derivative of $f_x(x, y)$ with respect to x is taken, the result is:

$$\frac{\partial f_x(x, y)}{\partial x} \approx \frac{\Delta f_x(x, y)}{\Delta x} \approx \frac{\Delta m_x}{\Delta x}$$

Geometrically, this is the rate at which the slope in the x direction is changing as we move in the x direction.

We can follow the same procedure to approximate $f_{xy}(x, y)$ that we followed in previous sections to determine $f_{xy}(0, 0)$ on the surface $f(x, y) = xy$. The slope of the tangent line in the x direction at the point $(0, 0, 0)$ is equal to zero. The slope of the tangent line in the x direction at the point $(0, 2, 0)$ on the surface $f(x, y) = xy$ is equal to two.



Correspondingly, we can conclude that

$$\frac{\partial f_x(x, y)}{\partial y} \approx \frac{\Delta f_x(x, y)}{\Delta y} \approx \frac{\Delta m_x}{\Delta y} = \frac{2}{2} = 1$$